Quark-Gluon Plasma as a Condensate of $Z(3)$ Wilson Lines

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Effective theories for the thermal Wilson line are constructed in an $SU(N)$ gauge theory at nonzero temperature. I propose that the order of the deconfining phase transition for $Z(N)$ Wilson lines is governed by the behavior of $SU(N)$ Wilson lines. In a mean field theory, the free energy in the deconfined phase is controlled by the condensate for $Z(N)$ Wilson lines. Numerical simulations on the lattice, and the mean field theory for $Z(3)$ Wilson lines, suggest that about any finite temperature transition in QCD, the dominant correlation length increases by a large, uniform factor, of order five.

A new phase of matter, the Quark-Gluon Plasma, might be produced in the collisions of large nuclei at very high energies. By asymptotic freedom, the pressure approaches the ideal gas value in the limit of high temperature, and so it is natural to think of the high temperature phase of QCD as a gas of quasiparticles [1,2].

It is known, however, that the *high* temperature phase of a purely glue theory is like the *low* temperature phase of a spin system. The magnetization in the high temperature phase of a $SU(N)$ gauge theory is a $Z(N)$ spin, proportional to the trace of the thermal Wilson line [3].

In this paper I construct effective lagrangians for the thermal Wilson line, considered as a full $SU(N)$ matrix, as well as its trace. This leads to novel sigma models of adjoint $SU(N)$ fields. Although the critical behavior is inexorably governed by the fixed point of $Z(N)$ spins [3], the $SU(N)$ spins can be important. In particular, they help explain why the order of the deconfining transition appears to change with N: from second order for $N = 2$ [4–6], to weakly first order for $N = 3$ [7–10], to first order for $N \geq 4$ [11,12]. Further, the picture of the high temperature phase is turned on its head: the pressure isn't due to quasiparticles [1,2], but is a potential for a condensate of $Z(N)$ Wilson lines. A mean field theory then suggests that because the deconfining transition in pure glue $SU(3)$ is weakly first order, QCD is near a critical point. About the transition, the dominant correlation lengths increase by a large factor, of order five [9].

I concentrate on the pure glue theory; later I argue why this is legitimate, using the lattice data and the effective theory. The thermal Wilson line is [3,13]

$$\mathbf{L}(x) = \mathcal{P} \exp\left(ig \int_0^{1/T} A_0(x,\tau)\, d\tau \right) \;, \qquad (1)$$

where $\mathcal{P}$ is path ordering, g is the gauge coupling constant, A_0 is the time component of the vector potential in the fundamental representation, x is the coordinate for three spatial dimensions, and τ that for euclidean time at a temperature T. The Wilson line in (1) is a product of $SU(N)$ matrices, and so is itself a $SU(N)$ matrix, satisfying

$$\mathbf{L}^\dagger(x)\,\mathbf{L}(x) = \mathbf{1} \quad , \quad \det(\mathbf{L}(x)) = 1 \qquad (2)$$

Without quarks, the allowed gauge transformations are periodic up to an element of a global $Z(N)$ symmetry [3]:

$$\mathbf{L}(x) \to \exp(2\pi i/N)\; \Omega^\dagger(x)\mathbf{L}(x)\Omega(x) \;, \qquad (3)$$

$\Omega(x) = \Omega(x,0)$. $\mathbf{L}(x)$ transforms as an adjoint field under local $SU(N)$ gauge transformations in three dimensions, and as a vector under global $Z(N)$ transformations.

Effective theories for $\mathbf{L}(x)$ are dictated by the symmetries of (3). I begin with the nonlinear form, where (2) are taken as constraints on $\mathbf{L}$. I construct an effective theory in three spatial dimensions, valid for distances $\gg 1/T$, by coupling the gauge potentials for static magnetic fields, the $A_i(x)$'s, to the Wilson line, $\mathbf{L}(x)$:

$$\mathcal{L}_0 \;=\; \frac{1}{2}\mathrm{tr}\left(G_{ij}^2\right) + T^2 a_1\, \mathrm{tr}|D_i\mathbf{L}|^2 \quad , \quad a_1 = \frac{1}{g^2} + \ldots \qquad (4)$$

The first term is the standard lagrangian for static A_i fields (by choice, A_i has dimensions of mass; all lagrangians have dimensions of (mass)^4). In the second term, I start with electric part of the gauge lagrangian, $\sim \mathrm{tr}|D_i A_0|^2$, and assume that it transmutes into a gauge invariant kinetic term for $\mathbf{L}(x)$. This is the continuum form of the lattice model of Banks and Ukawa [14].

Notice the factor of T^2 in front of the kinetic term for $\mathbf{L}$. This arises because the Wilson line is a phase in color space, and so every element is a dimensionless pure number. Thus in any effective lagrangian, dimensions can only be made up by powers of the temperature T.

Consider the somewhat peculiar limit in which one drops the coupling to the A_i's, by taking $g \to 0$, but retains $\mathbf{L} \neq \mathbf{1}$. Then (4) reduces a nonlinear sigma model in three dimensions, with lagrangian $\sim \mathrm{tr}|\partial_i\mathbf{L}|^2$. With the constraints of (2), the theory is invariant under $\mathbf{L} \to \Omega_1\mathbf{L}\Omega_2$, where Ω_1 and Ω_2 are independent, constant $SU(N)$ matrices. This is an enhanced global symmetry of $SU(N) \times SU(N)$ (times the usual global $Z(N)$ symmetry). As a sigma model, it is possible to impose other constraints upon $\mathbf{L}$ beyond those of (2). For example,

requiring tr$\mathbf{L}$ to be some fixed number produces a sigma model on a symmetric space [15].

At nonzero coupling, $\mathbf{L}$ is simply an adjoint field under the local $SU(N)$ symmetry. Even with the constraints of (2), the reduced symmetry implies that *many* more terms arise: instead of tr$\mathbf{L}$ being fixed, as for a symmetric space, arbitrary traces, such as tr$\mathbf{L}^p$ for integer p, are allowed. Mathematically, tr$\mathbf{L}^p$ is related to the trace of the Wilson line in higher representations [16].

At one loop order, the terms up to fourth order in A_0 have been computed. The quadratic term, $\sim \mathrm{tr}(A_0^2)$ [17,18], is the Debye mass for the gluon. For $N \geq 4$, there are two independent quartic terms, $\sim (\mathrm{tr}(A_0^2))^2$ and $\sim \mathrm{tr}(A_0^4)$ [19,20], which represent a potential for A_0. From (1), it is easy to turn a potential for A_0 into one for $\mathbf{L}$:

$$\mathcal{L}_1 = T^4 \left(c_2 |\mathrm{tr}\mathbf{L}|^2 + c_4 |\mathrm{tr}\mathbf{L}^2|^2 + c_4' \left(|\mathrm{tr}\mathbf{L}|^2 \right)^2 \right) \ . \tag{5}$$

Expanding to $\sim A_0^4$ fixes $c_2 = -(4+3/\pi^2)/9$, $c_4 = +(1+3/\pi^2)/36$, and $c_4' = 0$. The rational terms in c_2 and c_4 are from the Debye mass, while those $\sim 1/\pi^2$ arise from quartic terms in the potential for A_0. Only two constants, c_2 and c_4, are needed to fit three terms in the A_0 potential. If N_f flavors of massless quarks are included, c_2 and c_4 change, while c_4' is then nonzero.

The signs of c_2 and c_4 are interesting. As $\mathbf{L} \sim -g^2 A_0^2$, a positive Debye mass corresponds to negative c_2. The coupling c_4 is like the quartic term in A_0, and so positive. Negative c_2 favors condensation in a direction in which $|\mathrm{tr}\mathbf{L}|^2$ is maximized. This happens when $\mathbf{L}$ is an element of the center [21],

$$\langle \mathbf{L} \rangle = \exp(2\pi i j/N)\ \ell_0\ \mathbf{1} \ , \tag{6}$$

$j = 0...(N-1)$. Different j are the usual N degenerate vacua of the broken $Z(N)$ global symmetry.

In (6) I introduce an expectation value, $\ell_0 = \langle \ell \rangle$, where ℓ is defined in (8). In perturbation theory, $\ell_0 = 1$, but ℓ_0 is a function of temperature; it vanishes at the critical temperature, T_c, and in the confined phase, for $T < T_c$.

I assume that in the deconfined phase, $T > T_c$, the stable vacuum is that which maximizes $|\mathrm{tr}\mathbf{L}|^2$, so that $\mathbf{L}$ condenses as in (6). An expectation value for a field in the fundamental representation always breaks the gauge symmetry, but uniquely for an adjoint field, a vacuum expectation value proportional to the unit matrix does *not*: (6) is invariant under arbitrary local gauge rotations. Similarly, the adjoint covariant derivative in (4) is $D_i \mathbf{L} = \partial_i \mathbf{L} - ig[A_i, \mathbf{L}]$, so with (6), the static magnetic gluons do not acquire a mass when $\mathbf{L}$ condenses, $\ell_0 \neq 0$. Thus (6) is the nonperturbative statement that electric screening does not generate screening for static magnetic fields [13,20,22].

The terms in (5) are invariant under a global symmetry of $U(1)$. There are also terms which reduce this $U(1)$ to $Z(N)$. For $N = 3$, the simplest examples include

$$\det \mathbf{L} + \mathrm{c.c.}\ ,\ (\mathrm{tr}\mathbf{L})^3 + \mathrm{c.c.}\ ,\ \mathrm{tr}\mathbf{L}\ (\mathrm{tr}\mathbf{L}^2) + \mathrm{c.c.}\ . \tag{7}$$

The first term, $\det \mathbf{L}$, is $SU(3) \times SU(3)$ symmetric, while the others are only $SU(3)$ symmetric.

There are also a wide variety of kinetic terms possible. These include $|\partial_i \mathrm{tr}\mathbf{L}|^2$, $|\partial_i \mathrm{tr}\mathbf{L}^2|^2$, and $|\mathrm{tr}\mathbf{L}|^2 \mathrm{tr}|D_i \mathbf{L}|^2$, amongst others. At one loop order, the kinetic term in (3) is renormalized, and terms such as these may be generated; present calculations cannot distinguish [23]. Even for $g = 0$, none of these new kinetic terms are invariant under $SU(N) \times SU(N)$.

The potential in (5) is only illustrative. In perturbation theory, one expands about $\mathbf{L} \sim \mathbf{1}$, which does not allow one to uniquely fix the coefficients of a potential for $\mathbf{L}$. Through numerical simulations, effective theories for A_0 [20], and those for $\mathbf{L}$, could be matched by comparing physical correlation lengths at an intermediate temperature scale, say at several times the critical temperature.

Now consider a point of second order transition, where $\ell_0(T) \to 0$. Then powers of $\mathbf{L}$ are suppressed, and it is sensible to construct a linear sigma model. This is done by introducing an "block spin" $\mathbf{L}$, formed by a gauge covariant average of $\mathbf{L}$ over some region of space [24]. Any $SU(N)$ matrix can be written as

$$\mathbf{L}(x) = \ell(x)\mathbf{1} + 2i\tilde{\ell}_a(x) t^a \ , \tag{8}$$

where t^a are the generators of $SU(N)$, $a = 1 \ldots (N^2-1)$. For general N, ℓ and $\tilde{\ell}_a$ are complex valued, and (2) imposes N^2+1 constraints.

I start with the case of two colors, which is special. Four constraints of (2) are satisfied in an especially simple manner: the imaginary parts of ℓ and $\tilde{\ell}_a$ vanish. This leaves one constraint, which is $\ell^2 + \tilde{\ell}_a^2 = 1$; thus ℓ and $\tilde{\ell}_a$ form a vector representation of $SU(2) \times SU(2) = O(4)$. After averaging, the constraint on the $O(4)$ norm is lost, as is typical in a linear model. Averaging still leaves ℓ and $\tilde{\ell}_a$ as real valued fields, though. Up to quartic order, the most general lagrangian is

$$\mathcal{L} \ = \ \frac{1}{2} \mathrm{tr}\left(G_{ij}^2 \right) \ + \ \frac{1}{2} (\partial_i \ell)^2 \ + \ \mathrm{tr}|D_i \tilde{\ell}|^2 \tag{9}$$

$$-m_1(\ell^2 + \tilde{\ell}_a^2) - m_2 \ell^2 + \lambda_1 (\ell^2 + \tilde{\ell}_a^2)^2 + \lambda_2\, \ell^4 + \lambda_3\, \ell^2\, \tilde{\ell}_a^2 \ .$$

The ℓ-field is a color singlet, while $\tilde{\ell}_a$ is an adjoint $SU(2)$ field. The terms $\sim m_1$ and λ_1 are $O(4)$ symmetric; with the kinetic terms, they correspond to the gauged nonlinear sigma model of (4). The other terms, $\sim m_2$, λ_2 and λ_3, correspond to the potential for the Wilson line in (5). A factor of T has been absorbed into the definition of ℓ and $\tilde{\ell}_a$.

When $N \geq 3$, the constraints of (2) are nonlinear. Since a sum of two special unitary matrices is not necessarily special unitary, the average $\mathbf{L}$ must be taken to be a complex $N \times N$ matrix. Thus $\mathbf{L}$ includes a complex valued, color singlet field, ℓ, which I call a $Z(N)$ spin, and a complex valued, color adjoint field, $\tilde{\ell}_a$, which I call a $SU(N)$ spin.

Linear models like (9) can be written down for $N \geq 3$, although there is a plethora of terms. At quartic order there is one term which is $O(2N^2)$ symmetric, $(\mathrm{tr}\mathbf{L}^\dagger\mathbf{L})^2$, another which is $SU(N) \times SU(N)$ symmetric, $\mathrm{tr}(\mathbf{L}^\dagger\mathbf{L})^2$, and terms which are only invariant under $SU(N)$, such as ($\tilde{\ell} \equiv \tilde{\ell}_a t_a$)

$$(|\ell|^2)^2 \ , \ \ell \, \mathrm{tr}(\tilde{\ell}^\dagger)^2 \ell + \text{c.c.} \ , \ \ell^2 \, \mathrm{tr}(\tilde{\ell}^\dagger)^2 + \text{c.c.} \ ,$$
$$(\mathrm{tr}\tilde{\ell}^\dagger \tilde{\ell})^2 \ , \ |\mathrm{tr}\tilde{\ell}^2|^2 \ , \ \mathrm{tr}(\tilde{\ell}^\dagger \tilde{\ell})^2 \ , \ \mathrm{tr}(\tilde{\ell}^\dagger)^2 \tilde{\ell}^2 \ . \tag{10}$$

These models give a qualitative picture of the deconfining phase transition: I assume that while only the $Z(N)$ ℓ-spins condense, $\langle \ell \rangle \equiv \ell_0 \neq 0$, that the transition is driven by the behavior of the $SU(N)$ $\tilde{\ell}_a$-spins. This picture is based on the nonlinear model: at weak coupling, (4) dominates other terms, such as (5), by $\sim 1/g^2$. Now certainly all coupling constants change with T, as can be seen from the temperature dependence of the Debye mass [18]. Nevertheless, I assume that the $SU(N)$ $\tilde{\ell}_a$-spins dominate right down to the point of the deconfining phase transition. The only purpose of terms such as (5) is to ensure that condensation which respects the local $SU(N)$ symmetry, (6), is favored.

For two colors, the influence of the $SU(2)$ $\tilde{\ell}_a$-spins on the $Z(2)$ ℓ-spins is subtle. Assume that only the $O(4)$ symmetric mass, m_1, changes. The phase transition in a gauged $SU(2)$ model is known from lattice studies of the electroweak phase transition [25]. I assume that one is always in an extreme "type-II" regime, so that the second order $O(4)$ transition of the model with $g = 0$ (the point B_2 of fig. (1) in [25]) is washed out by confinement of nonabelian gauge fields. The only transition is a point at which the $Z(2)$ ℓ-spins become massless; the $SU(2)$ $\tilde{\ell}_a$-spins are always massive. Lattice studies confirm a second order transition in the $Z(2)$ universality class [5].

For $N \geq 3$, the $SU(N)$ $\tilde{\ell}_a$-spins can have first order transitions. This is because in the absence of gauge fields, $SU(N) \times SU(N)$ spin models have first order transitions both in mean field theory [26] and in an expansion about $4-\epsilon$ dimensions [25]. As suggested in [25], in the extreme type-II regime, confinement of the gauge fields need not wash out the first order transition of $SU(N) \times SU(N)$ spins (above the point B_3 in fig. (2) of [25]), and so the deconfining transition can remain first order. In particular, the transition can be of first order as $N \to \infty$. This is in accord with a lattice analysis of Gocksch and Neri [27,28], and contrary to previous speculation [11,30].

For three colors, this implies that the deconfining transition is of first order not only because of cubic invariants [3], as in (7), but because of the dynamics of $SU(3)$ $\tilde{\ell}_a$-spins. Relative to the ideal gas, the latent heat for three colors is $\sim 1/3$ [7]. As could have been guessed from the lattice data alone, perhaps the deconfining transition is weakly first order for $N = 3$ because it is near the second order transition for $N = 2$. Thus it is of value to know how the latent heat for $N = 4$ compares to that for $N = 3$: is it more strongly first order, such as $\sim (N-2)/N$ as $N \to 2$, or more weakly first order, like $\sim 1/N$ as $N \to \infty$ [11]?

Whatever the order of the deconfining phase transition, one can write a mean field theory in which the free energy in the deconfined phase is controlled by a potential for the $Z(N)$ Wilson lines. For three colors, this is [3,11]:

$$\mathcal{V} \ = \ \left(-2b_2 \, |\ell|^2 + b_3(\ell^3 + (\ell^*)^3) + (|\ell|^2)^2\right) \ b_4 \, T^4 \ . \tag{11}$$

ℓ is complex valued, so when $b_3 \neq 0$, the global symmetry is reduced from $O(2)$ to $Z(3)$. The coupling b_3 must be small for the transition to be weakly first order [31], so for now I ignore it, considering the potential just as a function of b_2 and b_4. This is similar to the case of two colors, where ℓ is a real field, and the potential is just a sum of two terms, $\sim b_2 \ell^2$ and $\sim b_4 \ell^4$ [32].

In speaking of the Wilson line, implicitly I assume that it is possible to extract a renormalized value [33] from the bare quantity [34]. If so, then given $\ell_0(T)$ and the pressure, one could fit to a potential like (11); for example, is it necessary to include higher powers of ℓ in $\mathcal{V}$?

The novel aspect of (11) is my insistence that because ℓ is a *dimensionless* field, the dimensions in $\mathcal{V}$ must be made up by the temperature, T. In mean field theory, b_4 is taken as constant, and b_2 varies with temperature, vanishing at T_c. The pressure is given by the minimum of the potential, $p = b_2^2 b_4 T^4$, and vanishes in the confined phase, $T < T_c$. That is, with the overall T^4 in the potential, the pressure is like a gas of quasiparticles, albeit with a variable number of degrees of freedom, which vanish at T_c.

At high temperature, $b_2 \to 1$ so that $\ell_0 \to 1$. The quartic coupling is fixed by the ideal gas limit: if $n_\infty = p/T^4$ as $T \to \infty$, $b_4 = n_\infty$. Lattice simulations [4–10] find that the p/T^4 is relatively flat down to a scale which is several times T_c, call it κT_c; the same is found from resummations of perturbation theory [2]. (κ might be defined as the lowest value where $\ell_0 \approx 1$.) Hence I assume that b_2 and b_4 are slowly varying down to κT_c.

Between κT_c and T_c, I assume that b_4 is essentially constant, while b_2 varies. In particular, the trace of the energy momentum tensor, divided by T^4, is $(e-3p)/T^4 = T\partial(b_2^2 b_4)/\partial T$. Lattice simulations find that this quantity has a peak just above T_c [4–10]: this is then due to the rapid variation of b_2 with temperature [32].

In the $Z(N)$ mean field theory, the pressure includes only the contribution of the potential, and nothing from fluctuations in the effective fields, either from the $Z(N)$ ℓ-spins or the $SU(N)$ $\tilde{\ell}_a$-spins. Fluctuations in these fields do, of course, contribute to the pressure at all temperatures. Since by construction the pressure in the mean field approximation vanishes for $T < T_c$, one condition for its validity is that the pressure in the confined phase is small. This is what present lattice simulations find. Physically, these fields don't contribute much to the pressure because they are heavy: the $SU(N)$ $\tilde{\ell}_a$-spins always so, and the $Z(N)$ ℓ-spins usually so. For two or three colors, the ℓ-spins do become light in a narrow band in temperature about T_c, where mean field theory fails [32].

Ignoring fluctuations about the mean field theory is also justified from the viewpoint of an expansion in a large number of colors, $N \to \infty$ [27,30,35]. The free energy in the confined phase is of order one, while it is $\sim N^2$ in the deconfined phase. The term $\sim N^2$ in the free energy is due *entirely* to the condensate, taking $b_2 \sim 1$ and $b_4 \sim N^2$. Even though there are $\sim N^2$ of them, the $SU(N)$ $\tilde{\ell}_a$-spins only contribute to the pressure at ~ 1, since they are bound into color singlet glueballs. The $Z(N)$ ℓ-spins also contribute ~ 1 to the free energy.

I have concentrated on the pure glue theory because numerical simulations have demonstrated the following remarkable property [7,8]. If $p/(n_\infty T^4)$ is plotted versus T/T_c, the resulting curve is nearly universal, and looks very similar whether or not there are dynamical quarks present. The present model predicts that the pressure is the same because the (renormalized [33]) Wilson line is the same. In terms of the potential, (11), the differences in the ideal gas values, n_∞, are absorbed into b_4, with the same $b_2(T/T_c)$.

Quarks act like a background magnetic field for the real part of ℓ [14,36]. Because the pure glue transition is weakly first order, it is not difficult for quarks to wipe out the deconfining transition altogether, leaving either a chiral transition, or just crossover behavior. Even so, what is relevant here is that for three colors and two or three flavors of quarks, the pressure for $T < T_c$ is always much smaller than that for $T > T_c$; that is, up, down, and strange quarks act like a *weak* magnetic field for the $Z(3)$ ℓ-spins.

I thus come to the central physical point of this paper. The lattice tells us that the deconfining transition in pure glue $SU(3)$ theory is close to the second order transition for $SU(2)$; further, that the effects of quarks are small, except close to T_c. I suggest that what is important is *not* whether the *weakly* first order transition persists with quarks, but that the *nearly* second order transition very well might. In the pure glue theory, as $T \to T_c^+$ the ratio of the screening mass to the temperature decreases by a factor of ten: from ~ 2.5 at $T \sim 2T_c$, to $\sim .25$ at $T \sim T_c^+$ [9]. Similarly, the string tension at $T \sim T_c^-$ is ten times smaller than that at zero temperature [9]. With quarks, the increase in the correlation length for ℓ is presumably less, maybe not ten, but perhaps a factor of five or so. And most importantly, if the pressure below T_c is small, it might be justified to use the $Z(3)$ mean field theory.

If the chiral order parameter is Ψ, then it couples to $Z(3)$ ℓ-spins through the coupling $+|\ell|^2 \mathrm{tr}\left(\Psi^\dagger \Psi\right)$. Lattice simulations find that the chiral and deconfining transitions occur at approximately the same temperature. This naturally results if this coupling constant is positive, as condensation in one field tends to suppress condensation in the other. Coherent oscillations in the ℓ-field couple to light mesons through such a term, and can produce large fluctuations in the average pion momentum [37].

This uniform increase in correlation lengths near T_c is a unique prediction of the $Z(3)$ mean field theory. In quasiparticle models of the quark-gluon plasma, the pressure is tuned to vanish at T_c by the introduction of a bag constant. In order for the energy to decrease as $T \to T_c^+$, though, the quasiparticles must become *heavier*, not lighter; that is, instead of increasing, most correlation lengths decrease [1,2].

At nonzero quark chemical potential, μ, presumably there is little change if the quarks are hot and dilute: for small μ/T, the $Z(3)$ ℓ-spins should still exhibit nearly second order behavior. I contrast this with the (possible) critical endpoint of the chiral transition in the $\mu - T$ plane [38]. The correlation length of the sigma meson truly diverges at the critical endpoint, but this only occurs at one special value of μ. Moreover, the sigma meson does not dominate the free energy, nor generic particle production. For cold, dense quark matter, $\mu \gg T$, I do not see why $Z(3)$ ℓ-spins should dominate the free energy.

I conclude by noting that the generalization of the Debye mass term, $\sim \mathrm{tr} A_0^2$, to real scattering processes produces hard thermal loops [39]. This is then the first term in an *infinite* series of such terms, continuing $\sim \mathrm{tr} A_0^4$, *etc.* The natural expansion is not in powers of A_0, but in powers of the Wilson line, as in (5). It is then of great interest to know the analytic continuation of the Wilson line to real scattering processes [40].

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[21] $\mathbf{L}$ can be diagonalized by a global gauge rotation. For a $U(N)$ matrix, $|\mathrm{tr}\mathbf{L}|^2$ is maximized when $\mathbf{L}$ is a constant phase times the unit matrix; for a $SU(N)$ matrix, then, $\mathbf{L}$ must be an element of the center.
[22] For discussion, consider what happens if the Debye mass is negative, so that the signs of c_2 and c_4 are switched, with $c_4 < 0$ and $c_2 > 0$. In this case, the favored vacuum maximizes $|\mathrm{tr}\mathbf{L}^2|^2$, with $\mathbf{L}^2$ an element of the center; then the vacuum might break $SU(N)$ to $SU(N-1)$.
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[24] To ensure that the average $SU(N)$ Wilson line transforms gauge covariantly, as in (3), the average is over "U" shaped Wilson paths: averaging over x', the path is spatial from $(x,0)$ to $(x',0)$, temporal up to $(x',1/T)$, and finally, spatial back to $(x,1/T)$.
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[28] As a function of N, the order of the deconfining transition in $SU(N)$ gauge theories appears to be similar to that for the N-state Potts models (F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982)). While $Z(N)$ ℓ-spins (trivially) interact like Potts spins for $N = 2$ and 3, they are not the same when $N \geq 4$ [29]. In mean field theory, though, Potts models are a special case of $SU(N)\times SU(N)$ models (R. G. Priest and T. C. Lubensky, Phys. Rev. **B13**, 4159 (1976)); thus I suggest that the proper view is that *both* deconfining and Potts transitions are driven by those of $SU(N)\times SU(N)$ spins.
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[31] b_3 is measurable from the two point function of the Wilson line near T_c. From (11), the mass squared for the real part of ℓ is $\sim \ell_0^2/a_1$, while that for the imaginary part is $\sim b_3\ell_0/a_1$, with a_1 from (4). Given ℓ_0, b_3 can be measured by comparing these two masses. This should be done not at high temperature, but close to T_c. At high temperature, $\mathbf{L}$ is nearly an $SU(N)$ matrix and $\ell_0 \approx 1$. The real and imaginary parts of ℓ satisfy a geometric constraint, and cannot be varied independently. The constraint is invalid near T_c, where the average $\mathbf{L}$ is not an $SU(N)$ matrix. $\mathcal{V}$ is applicable for real ℓ (or $Z(3)$ transforms thereof) at all T.
[32] Mean field theory fails close to a second order transition, but the effect of nontrivial critical exponents can be included, as in (12) of [4]. The critical point for two colors is $Z(2)$, while three colors is near an $O(2)$ fixed point.
[33] The renormalized $SU(N)$ Wilson line is given after extracting an overall renormalization constant. This constant is fixed by setting ℓ_0 at some temperature, such as $\ell_0 \to 1$ as $T \to \infty$; E. Gava and R. Jengo, Phys. Lett. **B105**, 285 (1981), [6,29], and A. V. Smilga, Phys. Rept. **291**, 1 (1997). In lattice perturbation theory, the ultraviolet divergence in the Wilson line is $\langle \mathbf{L}^p \rangle \sim -p^2 g^2 \langle A_0^2 \rangle \sim \mathbf{1}\exp(-p^2 g^2 d/(aT))$ after exponentiation, for some constant d and lattice spacing a. To isolate this divergence, it might be useful to measure the distribution of eigenvalues for the $SU(N)$ Wilson line, and then compute the trace from that distribution.
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[40] Define null vectors $K = (i,\hat{k})$ and $K_* = (-i,\hat{k})$, $\hat{k}^2 = 1$, and gauge "potentials",

$$\mathcal{A}^\alpha = \frac{1}{D\cdot K} K^\mu G^{\mu\alpha} \quad , \quad \mathcal{A}_*^\alpha = \frac{1}{D\cdot K_*} K_*^\mu G^{\mu\alpha} .$$

With $m_g^2 = (N + N_f/2)g^2T^2/3$ the Debye mass squared, and $\int d\Omega_{\hat{k}}$ the normalized integral over $\hat{k}$, the usual lagrangian for hard thermal loops is $\mathcal{L}_{htl} = m_g^2 \int d\Omega_{\hat{k}} \mathrm{tr}(\mathcal{A}_\alpha)^2$ [39]. This does not look like the expansion of a Wilson line, though. However, an equivalent form is

$$\mathcal{L}_{htl} = m_g^2 \int d\Omega_{\hat{k}} \, \mathrm{tr}\left(\mathcal{A}\cdot K_* \; \mathcal{A}_*\cdot K\right) .$$

which does look like an expansion of exponentials of $\mathcal{A}\cdot K_*$ and $\mathcal{A}_*\cdot K$.